On the Theory of Simple or Cruciform Wing and Wing-Body System over a wide range of Supersonic–Hypersonic Regime

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Summary

Considering various lifting systems, namely: simple or cruciform wing and wing-body system, the author presents the theory of such wing systems in supersonic regime, beginning with the first approximation based on the linearisation of equations (and hence, the assumption of small disturbances). Further, the author deals with the phenomenon of the flow separation at the leading edges, corresponding to thin wings with subsonic edges. Some theoretical considerations are made on wings with curved leading edges. In the case of wings with supersonic edges, the author also extends the theoretical considerations to the moderate-hypersonic regime. Theoretical and experimental values are compared.

Symbols

U_∞ free stream velocity

a_∞ speed of sound in the free stream

$M_\infty=\dfrac{U_\infty}{a_\infty}$ Mach number of the free stream

$B=\sqrt{(M_\infty^2-1)}=\cot\mu$

μ semi-angle of the Mach cone

t time

x_1, x_2, x_3 cartesian co-ordinates

y, z co-ordinates in the physical plane $x_1=1$, defined by relations (3)

$\mathfrak{y}, \mathfrak{z}$ co-ordinates in the auxiliary plane, defined by relations (5)

$\mathfrak{x}=\mathfrak{y}+\mathfrak{z}$ complex variable

X auxiliary plane defined by relation (15) or (18)
ξ, β co-ordinates of a point situated on the curved leading edge of the wing
$\phi(x_1, x_2, x_3, t)$ velocity potential
u, v, w components of the disturbance velocity upon Ox_1, Ox_2, Ox_3
$\mathscr{U}_{n-1}(\mathfrak{r})$ axial disturbance velocity function
$A_{nq}, C_{nq}, D_{nq}, G_{nq}, H_{nq}, A, C$ constants included in the expression of the axial disturbance velocity
τ stream deflection with respect to the direction of the free stream
α incidence of the undisturbed stream U_∞ with respect to the plane Ox_1x_2
β incidence of the undisturbed stream U_∞ with respect to the plane Ox_1x_3
χ yaw angle
$l = \cot \chi$ dimensionless semi-span of the triangular wing
C_p pressure coefficient
C_l lift coefficient
C_d wave drag coefficient
γ ratio of specific heats
K unitary similarity parameter defined by relation (36)

1. Introduction

Modern supersonic aircraft existing at present cover a wide range of incidence and Mach number; the aircraft under study and construction or experimentation can reach considerable Mach numbers.

In the following we shall try to include in a unitary presentation the conditions of motion over this whole range of incidence and Mach number, determining the pressures and the overall forces for various systems of lifting wings.

These lifting systems can be either simple or complex, i.e., wing alone, cruciform wing, wing-body system etc.; we shall first approach the study of the simple wing which constitutes the starting point of all applications and then extend the results obtained to more complex systems.

In view of the complicated geometry of the surface of the aircraft wing, it is impossible to obtain a rigorous and general solution of the problem without preliminary approximations. Consequently we shall proceed by stages, first considering the wing thickness and its variation over the whole surface as well as the wing incidence with respect to the free stream U_∞ to be sufficiently small in order to neglect certain secondary order terms in the general equations and reach, through their linearisation, the *theory of small disturbances*, which, as is known, has given remarkable results surprisingly well confirmed by tests, of course within the limits of this simplifying assumption.

The first stage of investigation, characterised by the method of small disturbances, cannot represent the actual phenomenon once the wing incidence exceeds a certain limit, even at small Mach numbers, or in the case when the wing thickness and incidence, as well as the Mach number of the free stream, assume large values. Consequently we have to take account of the effect of certain special phenomena which appear under these conditions, as for instance, the *flow separation* at the subsonic leading edges, as well as the flow conditions in *moderate-hypersonic regime*, where certain terms considered to be secondary ones in the first stage, now become important.

Other extensions, as for instance Newton's impact theory or the theory of ionized and rarified gases, are not included in our paper.

In order to establish better the application limits of the various theories and formulae, we give below an indicative scale of motion ranges in terms of the classic similarity parameter $K = M_\infty \tau$, where M_∞ is the Mach number of the free stream ($M_\infty > 1$) and τ is the flow deflection with respect to the initial direction:

$M_\infty \tau \ll 1$, the regime of small disturbances,
$M_\infty \tau \sim 1-2$, the moderate-hypersonic regime.

Although this delimitation is only indicative, we shall try to substantiate theoretically all stages that appear around the wing over a wide range of incidences and Mach numbers, giving the following succession and aspects of the motion:

— supersonic regime, under the assumption of small disturbances, at small incidences and Mach numbers;

— supersonic regime for thin wings, at any incidence but small Mach numbers, with flow separation at the subsonic leading edges;

– effects of curved edges;

— supersonic–hypersonic regime, at large incidences and Mach numbers.

All these aspects are founded on the same basic theoretical considerations, as it will be shown below.

2. Wing Theory Under the Assumption of Small Disturbances

It is known that, under the assumption of small disturbances, by making the allowed approximations, the motion equations lead to the following linearised equation of the velocity potential which will be denoted by

$\phi(x_1, x_2, x_3, t)$ and which is a function of the co-ordinates x_1, x_2, x_3, and of the time t:

$$-B^2\frac{\partial^2\phi}{\partial x_1^2}+\frac{\partial^2\phi}{\partial x_2^2}+\frac{\partial^2\phi}{\partial x_3^2}=\frac{1}{a_\infty^2}\left(\frac{\partial^2\phi}{\partial t^2}+2U_\infty\frac{\partial^2\phi}{\partial x_1\partial t}\right) \tag{1}$$

where a_∞ is the speed of sound under the conditions of the free stream, while B is given by the relation

$$B=\sqrt{(M_\infty^2-1)}=\cot\mu \tag{2}$$

μ being termed the Mach angle.

In the *steady motion*, the second member of the above equation is equal to zero and for certain wing shapes we can easily obtain the respective solutions.

It can be asserted that all problems arising in the case of moderately thick wings have been practically solved. Indeed, the linearised equation (1) allows us to separate the effects of the thin wing which form its frame, from those of the wing of symmetrical thickness around this frame, which together constitute the wing proper.

As mentioned above, all practical problems arising in the case of wings of any planform and any variation of the normal dimensions can be solved through the method of doublets and sources. Proceeding in this manner one encounters difficulties only in the effective calculus and laborious integrals.

By means of the method of homogeneous functions initiated by Beshkine(1) and Germain(19) and developed under various aspects in various countries and also at the Institute of Fluid Mechanics in Bucharest, one can define the *conical motions* around the wing, which imply however the condition that the leading edges be straight. Nevertheless, through the extension made by us to wings with curved leading edges, this latter method can now cover any planforms used for modern supersonic aircraft.

Let us refer the wing to the three rectangular axis system $Ox_1x_2x_3$; further, considering the wing to be practically contained in the plane Ox_1x_2 (Fig. 1*a*) and introducing the notations

$$\frac{x_2}{x_1}=y, \qquad \frac{x_3}{x_1}=z \tag{3}$$

which will define a plane Oyz, termed *physical plane* (Fig. 1*b*), normal to the axis Ox_1, the components of the *disturbance velocity*, u, v, w, will be set in terms of x_1, y, z. We shall particularly study the construction of the *axial disturbance velocity* u, since this one is explicitly included in the formula of the pressure coefficient:

$$C_p=-2\frac{u}{U_\infty} \tag{4}$$

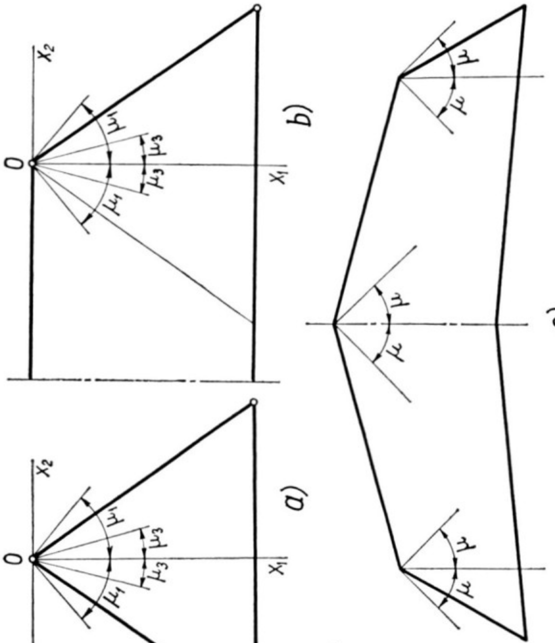

FIG. 2 — Different wing shapes. (*a*) Delta wing. (*b*) Trapezoidal wing. (*c*) Polygonal wing

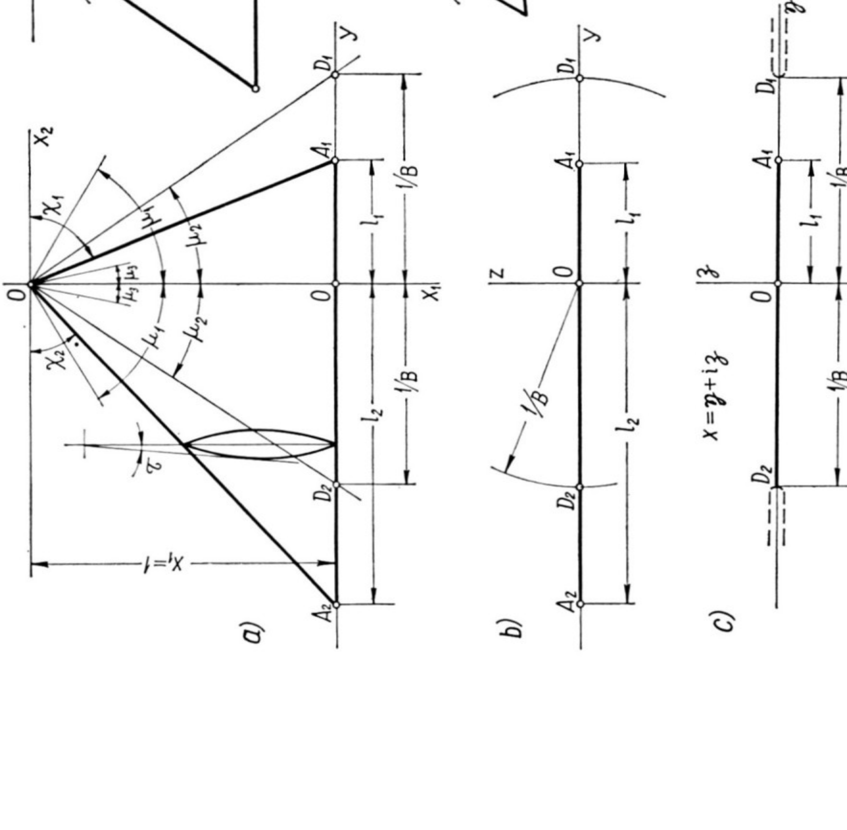

FIG. 1 — Triangular wing. (*a*) Wing planform. (*b*) Physical plane ($x_1=1$). (*c*) Auxiliary plane $\mathfrak{z}$

The solution of u in terms of x_1, y, z, is not easily obtained; therefore we introduce a change of variables, similar to that of Busemann, defined by the relations

$$\mathfrak{y} = \frac{y}{1-B^2z^2}, \quad \mathfrak{z} = \frac{z\sqrt{[1-B^2(y^2+z^2)]}}{1-B^2z^2} \tag{5}$$

which define an auxiliary plane $O\mathfrak{y}\mathfrak{z}$ (Fig. 1c). In this plane, the axial disturbance velocity will be set in the form

$$u = x_1^{n-1}u_{n-1}(\mathfrak{y}, \mathfrak{z}) = x_1^{n-1}\,\mathrm{Re}\,\mathscr{U}_{n-1}(\mathfrak{x}) \tag{6}$$

in which $u_{n-1}(\mathfrak{y}, \mathfrak{z})$ is a harmonic function representing the real part of the analytical function $\mathscr{U}_{n-1}(\mathfrak{x})$, termed *axial velocity function*, of complex variable

$$\mathfrak{x} = \mathfrak{y} + i\mathfrak{z} \tag{7}$$

In a section x_2=const, the angle τ made by the tangent to the profile in this section with the undisturbed velocity U_∞, represents the flow deflection, while the downwash on the wing, which also represents one of the boundary conditions, can be expressed by a homogeneous polynomial of order $(n-1)$:

$$w = \tau U_\infty = \sum_{q=0}^{n-1} w_{n-1-q,q}x_1^{n-1-q}x_2^q = x_1^{n-1}\sum_{q=0}^{n-1} w_{n-1-q,q}y^q \tag{8}$$

in which $w_{n-1-q,\,q}$ are constant coefficients. In terms of these coefficients, as well as of parameters l_1 and l_2 (Fig. 1),

$$l_1 = \cot\chi_1, \quad l_2 = \cot\chi_2 \tag{9}$$

the expression of $\mathscr{U}_{n-1}$ is accurately determined for all three positions of the wing with respect to the Mach cone (Fig. 1) and it can be put in the following general form which will be later considered for other extensions of this theory

$$\mathscr{U}_{n-1} = \mathscr{U}_{n-1}(\mathfrak{x}, l_1, l_2) \tag{10}$$

In our earlier paper we have given these general expressions for all ordinary polygonal forms, triangular wings, trapezoidal wings or delta wings, all of which derive from the following formulae:

$$\mathscr{U}_{n-1} = \frac{1}{\sqrt{[(l_1-\mathfrak{x})(l_2+\mathfrak{x})]}}\sum_{q=0}^{n} A_{nq}\mathfrak{x}^q \tag{11}$$

corresponding to the thin triangular wing having both leading edges subsonic (Fig. 1a, $\mu=\mu_1$), from which the delta wing also results by putting $l_1=l_2=l$ (Fig. 2a);

$$\mathscr{U}_{n-1} = \sqrt{\left[\frac{1+B\mathfrak{x}}{B(l_1-\mathfrak{x})}\right]}\sum_{q=0}^{n-1} A_{nq}\mathfrak{x}^q + \cos^{-1}\sqrt{\left[\frac{(l_1+l_2)(1+B\mathfrak{x})}{(1+Bl_1)(l_2+\mathfrak{x})}\right]}\sum_{q=0}^{n-1} H_{nq}\mathfrak{x}^q \tag{12}$$

corresponding to the thin wing having one subsonic leading edge and one supersonic (Fig. 1*a*, $\mu=\mu_2$), from which the trapezoidal wing is also obtained by setting $l_2 \to \infty$ (Fig. 2*b*);

$$\mathscr{U}_{n-1} = \cosh^{-1}\sqrt{\left[\frac{(1+Bl_1)(1-B\mathfrak{x})}{2B(l_1-\mathfrak{x})}\right]}\sum_{q=0}^{n-1} C_{nq}\mathfrak{x}^q + \cosh^{-1}\sqrt{\left[\frac{(1+Bl_2)(1+B\mathfrak{x})}{2B(l_2+\mathfrak{x})}\right]}\sum_{q=0}^{n-1} G_{nq}\mathfrak{x}^q + \sqrt{\left[\frac{1-B^2\mathfrak{x}^2}{B^2}\right]}\sum_{q=0}^{n-2} D_{nq}\mathfrak{x}^q \tag{13}$$

corresponding to the wing of symmetrical thickness, from which the thin wing with supersonic leading edges also derives, by setting $Bl_1>1$ and $Bl_2>1$ (Fig. 1*a*, $\mu=\mu_3$).

It should be noticed that all these formulae can also be appiled to a simple delta wing when it is yawed, in which case it can have all the three positions specified in Fig. 1*a*. As a matter of fact, we can apply one or several of the above formulae to any polygonal wing (Fig. 2*c*).

In a similar manner we can find analogous expressions for the downwash w, when the pressures (or the axial velocity u) are given. From the expression of w we deduce the wing surface. In our papers we have termed this operation *indirect problem*.

The above results can be applied similarly to the harmonic oscillating motions of low frequency of a wing. In such a case, the potential $\phi\ (x_1, x_2, x_3, t)$ can be in the form

$$\phi(x_1,x_2,x_3,t) = U_\infty e^{i(\omega t+kx_1)}\varphi(x_1,x_2,x_3), \quad \left(k = -\frac{\omega}{U_\infty}\frac{1+B^2}{B^2}\right) \tag{14}$$

where $\varphi(x_1, x_2, x_3)$ is independent of time and is called *reduced potential*. This one behaves like a potential in the steady motion, since the solutions are prefectly well known, having been deduced from this motion for which all current cases are indicated above.

3. Considerations on the Theory of Complex Lifting Systems

The wing theory based on the assumption of small disturbances can be applied easily to the complex lifting systems which are in current use in the construction of modern supersonic aircraft and rockets. Thus we can mention the cruciform wing (Fig. 3*a*) or tail (Fig. 3*b*), which derive from the first one as well as wing-body systems (Fig. 3*c*) or cruciform wing-body (Fig. 3*d*). We shall not make particular enlargements on these problems but, in order

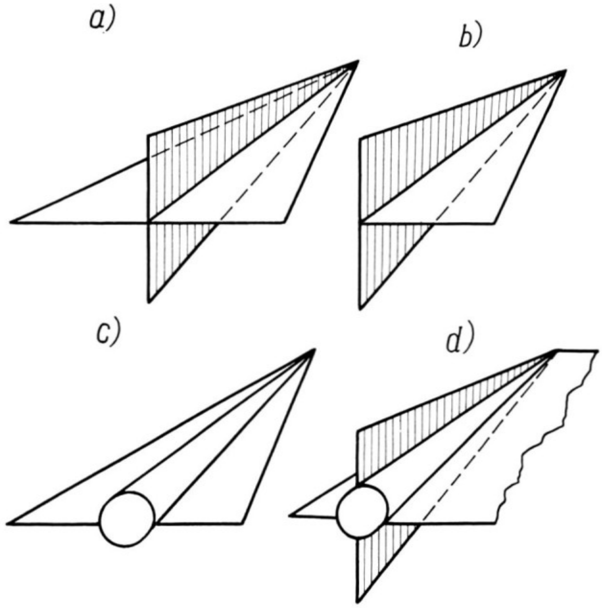

FIG. 3 — Different complex lifting systems. (*a*) Cruciform wing. (*b*) Horizontal and vertical tail. (*c*) Wing-body system. (*d*) Cruciform wing-body system

to illustrate the unitary method we have employed, we shall give below several simple examples.

Cruciform wing

Consider the case of the symmetrical cruciform wing, where the wing proper and the normal plate have constant incidences, two by two being equal and of opposite sense, the wing having the incidence α on the right-hand side portion and $-\alpha$ on the left-hand side one, and the plate having the lateral incidence, β on the upper portion and $-\beta$ on the lower one. The wing is defined by the length l, and the plate by the height h (Fig. 4*a*).

For the study of this wing system we have used the same transformation (5) as for the wing alone, which leads to the configuration given in Fig. 4b, as well as to an additional conformal transformation:

$$X^2 = \mathfrak{x}^2 + \mathfrak{h}^2, \quad \left(\mathfrak{h}^2 = \frac{h^2}{1-B^2h^2}\right) \tag{15}$$

which reduces the cruciform wing to a wing alone in the new plane (Fig. 4*c*).

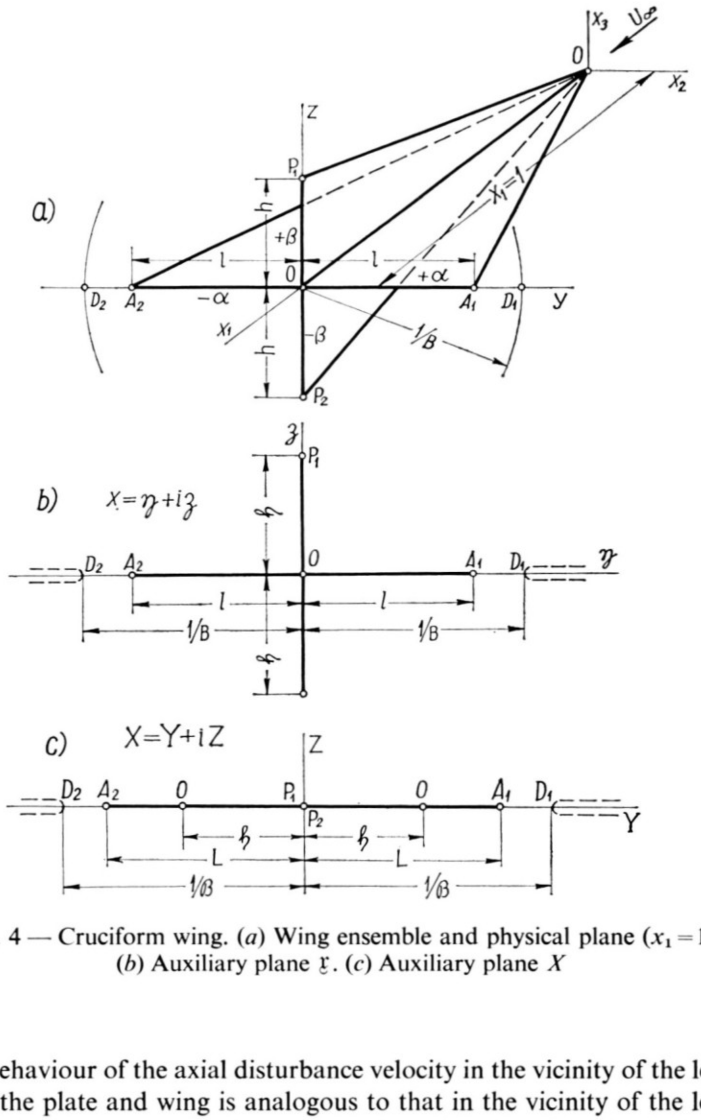

FIG. 4 — Cruciform wing. (*a*) Wing ensemble and physical plane ($x_1 = 1$). (*b*) Auxiliary plane $\mathfrak{x}$. (*c*) Auxiliary plane X

The behaviour of the axial disturbance velocity in the vicinity of the leading edge of the plate and wing is analogous to that in the vicinity of the leading edges of the wing alone. Further one can show that the axial velocity function $\mathscr{U}$ is real on the wing and on the plate.

In the X-plane, the calculations are similar to those performed in the case of a wing alone and, therefore, the solution is easily to be found. If we then return to the initial plane $\mathfrak{x}$, we find for the axial velocity,

$$\mathscr{U} = \frac{A_0 + A_2\mathfrak{x}^2}{\sqrt{[(l^2 - \mathfrak{x}^2)(\mathfrak{h}^2 + \mathfrak{x}^2)]}} \tag{16}$$

in the case when the leading edges of the wing and plate are subsonic ($Bl<1$, $Bh<1$), and

$$\mathscr{U} = A\sqrt{\left[\frac{1-B^2\mathfrak{x}^2}{\mathfrak{h}^2+\mathfrak{x}^2}\right]} + C\cos^{-1}\sqrt{\left[\frac{(l^2+\mathfrak{h}^2)(1-B^2\mathfrak{x}^2)}{(1+B\mathfrak{h}^2)(l^2-\mathfrak{x}^2)}\right]} \tag{17}$$

in the case when the leading edges of the wing are supersonic ($Bl>1$), while those of the plate are subsonic ($Bh<1$). The constants in these relations are determined in terms of α, β, l, h, according to the known patterns. The cases when the wing and plate have more complicated variations of incidence can be approached in a similar manner, but the restricted space prevents us from enlarging upon this problem. We have wanted only to point out the fact that on the basis of the same theory and unitary method, one can also find the solutions of this more complex problem.

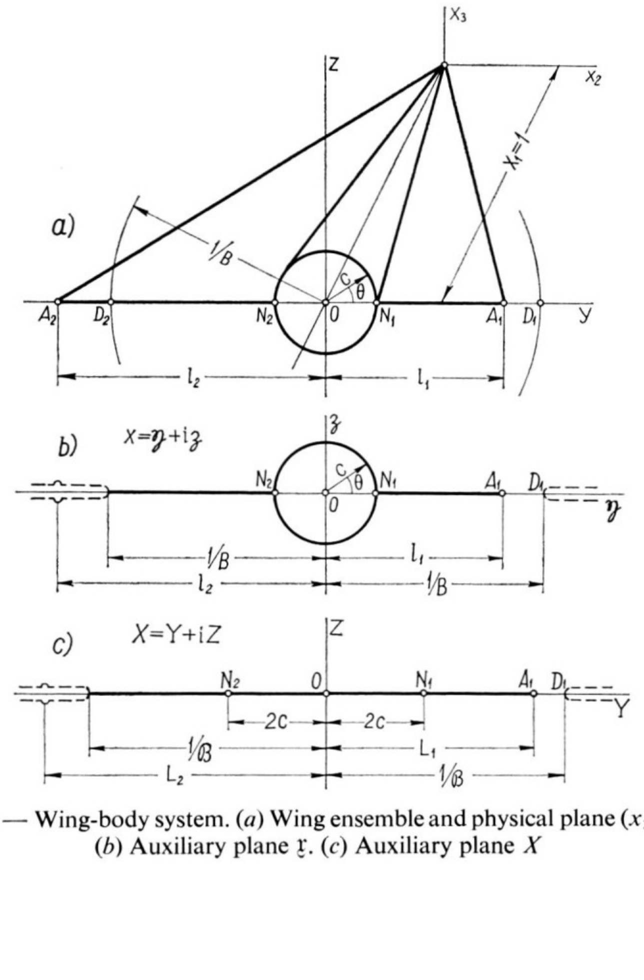

FIG. 5 — Wing-body system. (*a*) Wing ensemble and physical plane ($x_1 = 1$). (*b*) Auxiliary plane $\mathfrak{x}$. (*c*) Auxiliary plane X

Wing-conical body system

Consider now, also for reasons of simplicity, the wing-conical body system given in Fig. 5*a*.

It is to be noticed that, under the assumption of small disturbances, from the flow around this complex system one can separate the axisymmetric flow around the body alone without incidence.

Dealing further with the remaining motion resulting from this separation, one can demonstrate that the axial velocity function is real on the wing and body. Establishing for this function the boundary conditions as well as the singularities entailed by it, the problem can be easily approached in an auxiliary plane X, defined, with respect to the $\mathfrak{x}$ plane (Fig. 5*b*), by the conformal transformation

$$X = \mathfrak{x} + \frac{c^2}{\mathfrak{x}} \tag{18}$$

in which the problem is being reduced to the study of a fictitious wing alone having conical variable incidence (Fig. 5*c*).

In this manner one can approach the various cases of the thin wing or of the wing of symmetrical thickness fitted with conical body. For exemplification we shall give, in the $\mathfrak{x}$ plane, the expression of the axial velocity in the simple case of a thin delta wing with subsonic edges ($Bl<1$), the wing having a different incidence from that of the body:

$$\begin{aligned}\mathcal{U} = A\,&\frac{1+(c^4/l^4)-2(c^4/l^2\mathfrak{x}^2)}{\sqrt{\{(l^2+\mathfrak{x}^2)[1-(c^4/l^2\mathfrak{x}^2)]\}}} \\ &-\frac{2cC}{\pi}\left(1-\frac{c}{\mathfrak{x}}\right)\cosh^{-1}\sqrt{\left[\frac{(l+c)^2(l-\mathfrak{x})(c^2+l\mathfrak{x})}{2l(l^2+c^2)(\mathfrak{x}-c)^2}\right]} \\ &-\frac{2cC}{\pi}\left(1+\frac{c}{\mathfrak{x}}\right)\cosh^{-1}\sqrt{\left[\frac{(l+c)^2(l+\mathfrak{x})(c^2+l\mathfrak{x})}{2l(l^2+c^2)(\mathfrak{x}+c)^2}\right]}-\frac{ic^2C}{\mathfrak{x}}\end{aligned} \tag{19}$$

where A and C are constants.

When the incidence of the body axis is equal to that of the wing, then $C=0$ and the above expression reduces to the first term in the right-hand side. We shall further remark that in the same manner one can approach more complex problems of wing–body systems; we confine ourselves however to this simple example, which has been given in order to indicate the method of approach in various applications.

4. Theory of Thin Delta Wings on Taking Account of the Flow Separation at the Subsonic Leading Edges

As long as the attack angle α of a flat delta wing (or, better, the ratio α/l) is small, the pressures on the wing abide, to a certain degree, by the laws established under the assumption of small disturbances, with the exception of the points lying in the immediate vicinity of the subsonic leading edges, where the actual velocity is however finite and not infinite, as it would result from the theory. This fact is due to the phenomenon of the flow separation at the subsonic leading edges of the thin wing.

Once the ratio α/l increases, the effect of this flow separation also increases, becomes more prominent and assumes the shape of a layer of vortices under the form of cornet, with a more concentrated nucleus at a certain point of abscissa c and ordinate t (Fig. 6*a*). In this problem M. Roy obtained many interesting results[33, 34]. The motion remains conical and it can be treated by using the same method of the conical motions. To this effect, an interesting study has been carried out with a good approximation for slender wings ($Bl \ll 1$) by Mangler and Schmidt[26], which, however, cannot be applied to larger values of Bl in the interval $0 < Bl < 1$.

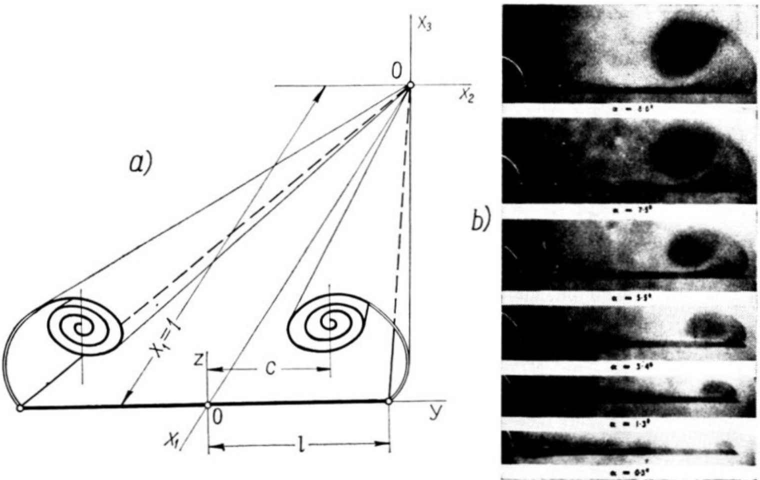

Fig. 6 — Thin delta wing with flow separation at the subsonic leading edges. (*a*) General sketch. (*b*) Photograph of cornets and vortex nucleus according to L. C. Squire[36] (with the permission of the Controller of Her Britannic Majesty's Stationery Office)

Consequently, on watching the scheme given in Fig. 6, instead of the actual motion due to this configuration of the vortex cornet, we shall rather consider its effect, namely, that of inducing an additional downwash, that is, an additional non-homogeneous vertical stream which alters the whole flow on the wing surface and hence, a finite velocity results at the edge. In fact the phenomenon is far more complex, but it can be schematised in its effects for simple computation.

Considering therefore the effect of vortices, the wing scheme under these simplifying conditions will be composed of the following three component wings, each of which can be treated separately, by applying the method used in the conical motions proper:

1. A thin wing having a *conical variation of incidence*, from which a *finite* velocity should result at the leading edges and which should render in a qualitative manner the effect of the vortex system on the upper surface. On the lower surface of this thin wing the axial disturbance velocity is equal and of opposite sense to that on the upper surface. The pressures are equal and of opposite sense on the two surfaces.

In order to avoid an infinite velocity at any point of the wing, the incidence variation at the wing extremity will be considered continuous and for the sake of simplicity, will be substituted by a corresponding source distribution which connects the source intensity q with the downwash w on the wing:

$$q = \frac{y}{\sqrt{(1-B^2y^2)}}\frac{\mathrm{d}w}{\mathrm{d}y} \tag{20}$$

This relation results from the method of the *hydrodynamic analogy* used in establishing the expression of $\mathscr{U}$ under a complex form. The above assumption results from the suggestion of Küchemann[(21)] and Squire [(35)] who use a sudden variation of incidence at the point $y = c$.

2. A wing of symmetrical thickness with the same slope variation as the incidence in the first case (1), hence with the same source distribution, which has the role of levelling the pressures on the lower surface, thus avoiding the pressures to be equal and of opposite sign on the two wing surfaces.

3. A wing of symmetrical thickness having a mean slope which should cancel the mean slope of the previous wing (2); for this reason we shall term it *compensatory wing*. In this manner, the resultant wing has the mean thickness equal to zero, as is the actual case. For this wing we shall consider a source distribution which should give an axial velocity varying slightly along the span, obtained through the expression:

$$q = k_0 y\left(1-\tfrac{1}{2}\frac{y}{l}\right) \tag{21}$$

Taking account of the boundary conditions, as well as of the u_l special conditions resulting from the effects of the vortex cornets, we obtain the axial velocities corresponding to the three wings: u_l for the wing (1), u_t for the wing (2) and u_c for the wing (3). Their sum yields the total velocity:

$$u = u_l + u_t + u_c \tag{22}$$

Because the calculations are laborious and the expressions of the axial velocities are lengthy, it is not the case to give either calculus details or the expressions of the velocities.

For calculating the lift, only u_l is to be taken into consideration; for calculating the pressures, the total velocity u has to be considered.

It is of interest to compare the theoretical results obtained in the above manner with the experimental ones. To this end, we have plotted in Fig. 7 the curve C_l in terms of the ratio α/l for various values of the parameter Bl. There is good agreement between theory and the tests carried out by O. A. Ormerod and A. B. Haynes[32] on the basis of overall measurements.

It is more interesting to compare the pressures (Fig. 8 and Fig. 9). Indeed,

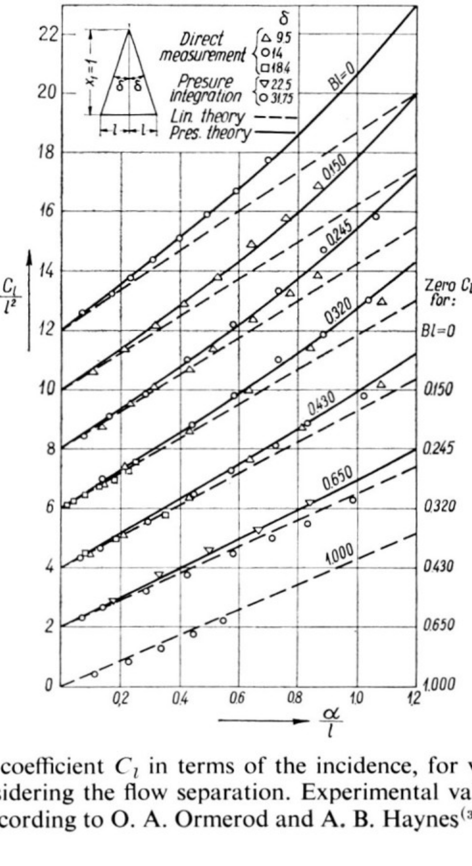

FIG. 7 — Lift coefficient C_l in terms of the incidence, for various values of Bl, on considering the flow separation. Experimental values are given according to O. A. Ormerod and A. B. Haynes[32]

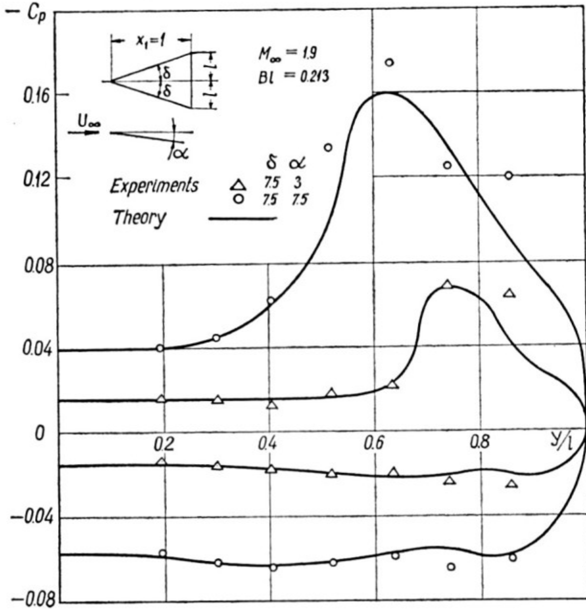

FIG. 8 — Pressure distribution on thin delta wing ($Bl = 0{\cdot}213$) with flow separation at the subsonic leading edges. Experimental values are given according to W. H. Michael

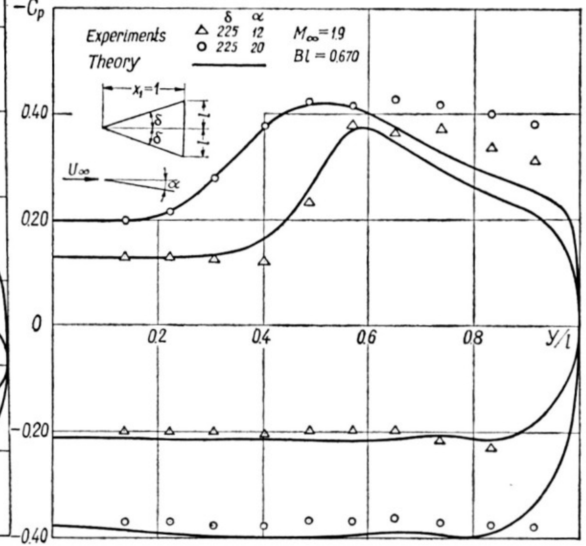

FIG. 9 — Pressure distribution on thin delta wing ($Bl = 0{\cdot}67$) with flow separation at the subsonic leading edges. Experimental values are given according to W. H. Michael[29]

the theoretical diagrams representing the pressure coefficient C_p according to formula (4) are in surprising agreement with the tests carried out by W. H. Michael[29].

All these results are related to the position of the vortex nuclei and particularly to the abscissa which has been denoted by c in Fig. 6*a*.

The scheme of the nucleus formation is given in the photograph (Fig. 6*b*) obtained by Squire[36].

Indeed, the vortex cornet turns into a main nucleus and, in another area, into a layer of vortices which remains on the cornet or turns into a small tip vortex. When the incidences are large, between the main nucleus and the small tip one, there appears another inboard nucleus, the sense of which is opposite to the former ones. A more profound analysis is not possible because of the complexity of this phenomenon. Consequently we shall consider only the simple scheme of Fig. 6*a*, assuming that the centre of the main nucleus has the abscissa $y=c$ on the wing trace in the plane Oyz. For the case $Bl \ll 1$, the accurate study of the problem[25, 26] shows that $c=l$ for $\alpha=0$ and that it varies inwards, according to a very complicated law; however, the distance c predicted by the theory differs from the one found experimentally. The disagreement is still greater for larger Bl $(0<Bl<1)$.

Consequently we shall use a simple reasoning in the general case. Thus, for instance, for very small incidences $(\alpha=0)$, the concentrated nucleus can be considered as adhering to the upper surface of the wing, a fact that would lead, no matter how small its intensity can be, to very high local velocities, incompatible with the actual effects of the flow separation. Therefore we have to admit that, for $\alpha=0$, we shall necessarily have $c=l$, in keeping with the theoretical findings for $Bl \ll 1$.

For large incidences, assuming that the vortex filaments detaching from the leading edge which are proportional to the wing breadth at these points, spin around an axis which represents the line of the weight centre defined by the abscissa

$$c = \tfrac{2}{3}l \tag{23}$$

We have noted above that the intensity of the vortex nucleus is somewhat smaller than the total intensity (approx. 80%), the remaining part being in the thin layer, which forms the cornet surface and gives rise to the tip vortex. This distance can vary in such a case about the value

$$c/l \approx 0{\cdot}6 \tag{24}$$

according to the approximate indications given by the tests carried out up to now, which are not however edifying, since the results differ by the authors and the test conditions.

Between these limit positions, by taking also account of the experimental indications, we can admit an approximate law:

$$\frac{c}{l} = \frac{1}{1+1{\cdot}7(\alpha)^{1/2}} \tag{25}$$

We have further assumed that the point of 'maximum depression' on the upper surface corresponds just to $y \approx c$, a fact that has made possible a convenient choice of the source distribution q of the first component wing defined above.

The method pointed out above can also be applied to other cases, as for instance to the thin delta wing in uniform rotation around the axis Ox_2, in which case however the conical motion is of order, $n=2$ a fact which also implies a corresponding method of approach, or to the trapezoidal wing with one subsonic edge etc.

5. Extension of the Theory to Wings With Curved Leading Edges

Although very general, the studies and results presented above are however confined to wings with straight leading edges. But, the planforms of wings employed in modern supersonic aviation generally have curved leading edges.

Consequently we shall introduce the notion of quasi-conical motions, with the aid of which we can extend the results previously obtained to this category of wings. To this effect, we shall proceed from the expressions of the axial disturbance velocities given above and find the value of the conical potential on the wing surface ($Z=0$) which will be denoted by ϕ_n. We shall apply the following formula which has been established in our previous papers[13]:

$$\phi_n = \operatorname{Re} \frac{x_2^n}{n} \int \mathscr{U}_{n-1} \, \mathrm{d}\left(\frac{1}{\underset{\smile}{y}^n}\right) = -x_2^n \int \frac{u_{n-1}}{y^{n+1}} \, \mathrm{d}y \tag{26}$$

This potential is function of x and of the geometrical parameters l_1 and l_2[9], which are constant for wings with straight leading edges.

By analogy with the lateral motion of fusiform slender bodies, where the motion is considered to be approximately plane in each of the sections of the slender bodies and, accordingly, one finds a quasi-plane potential which varies along the axis Ox_1, we shall also define in our case a quasi-conical potential termed like this owing to the variation of l_1 and l_2 in terms of x_1, a potential that will be denoted by ϕ^*. The quasi-conical axial velocity derives from this potential:

$$u^* = \frac{\partial \phi^*}{\partial x_1} \tag{27}$$

Let us now perform an application to the flat triangular wing with subsonic edges (Fig. 10*a*). For the conical motion proper ($n=1$), the axial disturbance velocity is given by the expression

$$\frac{u}{\alpha U_\infty} = \frac{2l_1 l_2 + (l_1 - l_2)y}{2E(k)\sqrt{[(l_1 - y)(l_2 + y)]}} \tag{28}$$

where $E(k)$ is the complete elliptical integral of the second kind having the modulus k. The corresponding conical potential is

$$\frac{1}{\alpha U_\infty}\phi_1 = \frac{\sqrt{[(b_1 - x_2)(b_2 + x_2)]}}{E(k)}, \quad (b_1 = x_1 l_1,\ b_2 = x_1 l_2) \tag{29}$$

If b_1 and b_2 do not vary linearly with x_1 (Fig. 10*b*), then the above conical potential becomes a quasi-conical potential ϕ^*, from which the quasi-conical axial velocity is resulting:

$$\frac{u^*}{\alpha U_\infty} = \frac{1}{\alpha U_\infty}\frac{\partial \phi_1^*}{\partial x_1} = \frac{b_2(\mathrm{d}b_1/\mathrm{d}x_1) + b_1(\mathrm{d}b_2/\mathrm{d}x_1) + x_2[(\mathrm{d}b_1/\mathrm{d}x_1) - (\mathrm{d}b_2/\mathrm{d}x_1)]}{2E\sqrt{[(b_1 - x_2)(b_2 + x_2)]}}$$
$$- \frac{\sqrt{[(b_1 - x_2)(b_2 + x_2)]}}{E^2}\frac{\mathrm{d}E}{\mathrm{d}x_1} \tag{30}$$

where E is now a constant which will be determined below.

It is further to be noticed that at an arbitrary point C_1 of co-ordinates x_1, x_2, the wing influence reduces to the portion delimited by the Mach lines C_1P_1 and C_1P_2. Taking account of the notations given in Fig. 10, we can show that the tangents to the lines of the leading edges will be

$$\frac{\mathrm{d}b_1}{\mathrm{d}x_1} \approx \frac{\mathrm{d}\beta_1}{\mathrm{d}\xi_1} = \lambda_1 = \cot\chi_1 \tag{31a}$$

$$\frac{\mathrm{d}b_2}{\mathrm{d}x_1} \approx \frac{\mathrm{d}\beta_2}{\mathrm{d}\xi_2} = \lambda_2 = \cot\chi_2 \tag{31b}$$

respectively.

With these notations, expression (30) becomes

$$\frac{u^*}{\alpha U_\infty} = \frac{l_2\lambda_1 + l_1\lambda_2 + (\lambda_1 - \lambda_2)y}{2E\sqrt{[(l_1 - y)(l_2 + y)]}} - \frac{x_1\sqrt{[(l_1 - y)(l_2 + y)]}}{E^2}\frac{\mathrm{d}E}{\mathrm{d}x_1} \tag{32}$$

By an appropriate method we can determine the constant E.

In order to calculate the values λ_1, λ_2, as well as

$$l_1 = \frac{\beta_1}{\xi_1}, \quad l_2 = \frac{\beta_2}{\xi_2} \tag{33}$$

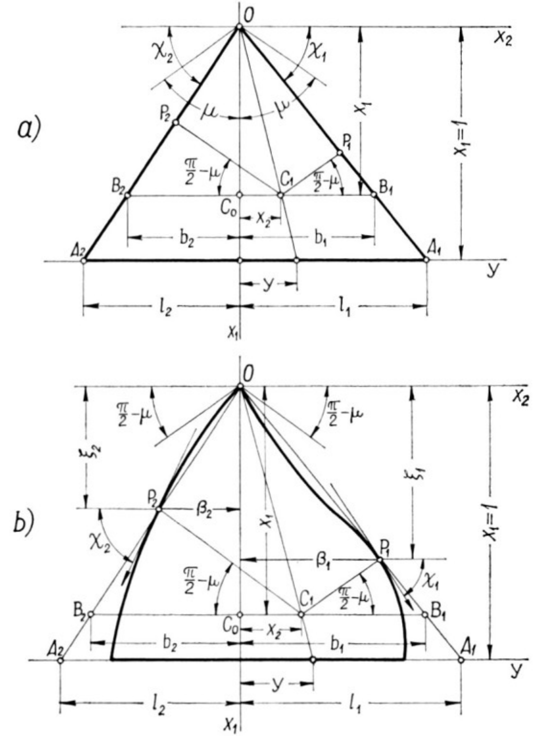

FIG. 10 — Triangular wing. (*a*) Triangular wing with straight leading edges. (*b*) Wing with curved leading edges. Gothic wing

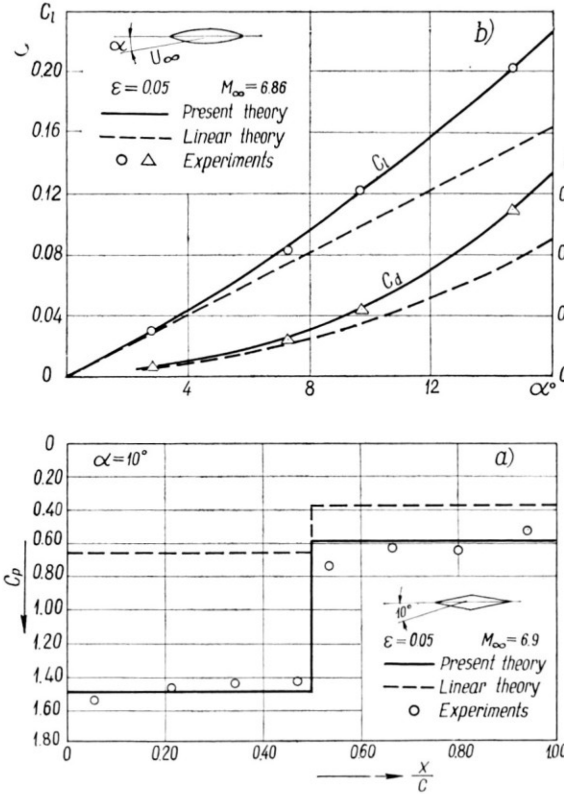

FIG. 11 — Experimental verifications. (*a*) Pressure distribution on a diamond-shaped profile, according to C. H. McLellan(28). (*b*) Lift and wave drag coefficient (C_l, C_d) according to Garrick, I. E.(18)

we have to know ξ_1 and ξ_2, which will be given by the expressions

$$\xi_1 + B\beta_1 = x_1 + Bx_2, \quad \xi_2 + B\beta_2 = x_1 - Bx_2 \tag{34}$$

With the aid of the above results we can pass to the calculation of pressures in any section $x_1 = \text{const}$ on 'ogee' shaped wings. Further, if we take also account of the complex wing-body system, the above theory allows us to approach the general problem of the wing with curved leading edges fitted with a body of arbitrary shape.

6. Aerodynamic Characteristics of Wings in Supersonic–Hypersonic Regime

For the moderate-hypersonic regime one can apply Lighthill's theory[26]. In yet another manner, the results obtained on the basis of the linearised theory applied to wings with supersonic leading edges can be extended to high regimes of the Mach number and great deflections of the undisturbed stream, in a word, they can be extended to the moderate-hypersonic regime. To this effect, we shall assume from the beginning that the stream deflection τ (or the dihedron angle of the wing leading edge) remains all the time in the framework of the shock waves attached to the wedge, and the Mach number M does not exceed an arbitrary limit over which the structure of the stream undergoes radical changes. In this case, we have shown in our previous papers that the pressure coefficient, deduced from the equations of the oblique shock waves, is given by the following formula

$$\frac{C_p}{m^2 \sin^2 \tau} = \frac{\gamma+1}{2} + \frac{2}{K}\sqrt{\left[1 + \left(\frac{\gamma+1}{4}\right)^2 K^2\right]} \tag{35}$$

in which K, that will be termed unitary similarity parameter, is given by the expression

$$K = M_\infty m \sin \tau, \quad \left(m = \frac{M_\infty}{\sqrt{(M_\infty^2 - 1)}}\right) \tag{36}$$

This formula is valid both for the supersonic and the moderate-hypersonic flow, therefore, over a wide range of incidences and Mach numbers. On the other hand, it can be shown that the same formula is accurately valid also in the case of expansion, where the values obtained are very close to those corresponding to the Prandtl–Meyer expansion, such that it can be asserted that (35) is a unitary formula for compression– expansion and for both regimes — supersonic and moderate-hypersonic. It is to be noticed that for very large values of K, (35) is reduced to Newton's formula, provided we obviously assume $\gamma \to 1$, a fact that can be easily deduced from the equations of the shock waves.

The experiments confirm this formula surprisingly well, such that we can consider it to be a basic formula for the subsequent developments. In Fig. 11, theory and experiments are compared[18, 28].

Thus, for instance, in the case of a leading edge which is sloped by the angle χ (as in the case of delta wings for example), the pressure coefficient is yielded by the following relation:

$$\frac{\lambda^2 C_p}{m^2 \sin^2 \tau} = \frac{\gamma+1}{2} + \frac{2\lambda}{K} \sqrt{\left[1+\left(\frac{\gamma+1^2}{4}\right)\frac{K^2}{\lambda^2}\right]} \tag{37}$$

in which

$$\lambda = \sqrt{(1-\tan^2\chi\tan^2\mu)} = \sqrt{[1-(1/B^2 l^2)]} \tag{38}$$

As is known however, the deflection τ is related to the axial disturbance velocity u through the relation

$$\frac{u}{U_\infty} = -\frac{\sin\tau}{B\lambda} \tag{39}$$

whence

$$-M_\infty^2 \frac{u}{U_\infty} = M_\infty \frac{m\sin\tau}{\lambda} = \frac{K}{\lambda} \tag{40}$$

Finally, formula (37) will be written in terms of u:

$$M_\infty^2 C_p = \frac{\gamma+1}{2}\left(M_\infty^2 \frac{u}{U_\infty}\right)^2 - 2\left(M_\infty^2 \frac{u}{U_\infty}\right)\sqrt{\left[1+\left(\frac{\gamma+1}{4}\right)^2\left(M_\infty^2 \frac{u}{U_\infty}\right)^2\right]} \tag{41}$$

We can apply the same reasoning as above to a wing constituted of a conical surface with the vertex at the origin and obtain, for each generatrix, the pressure coefficient given by the above formula in terms of the axial disturbance velocity u, which is obtained on the basis of the linearised theory.

We shall further consider that formula (41) is valid for any conical motion of order one or of a higher order, for which u is easily determined by means of the ordinary methods. Indeed, at a point on the leading edge, the local motion conditions are the same as those used for establishing formula (37). Both in the case of the compression behind the shock wave, due to the positive dihedron of the leading edge, and in the case of expansion, when the deflection upon the leading edge is negative, formula (41) records this variation in an accurate manner. After this, the stream undergoes small changes which can be considered isentropic, such that this formula is valid even in the interior of the Mach cone. The experimental verifications are edifying in this respect.

Thus, we have reproduced in Fig. 12 the experiments carried out by Wilby[40] on a wing having the slope constant also at $M_\infty = 7{\cdot}35$, experiments which are in good agreement with the theoretical data in which account has

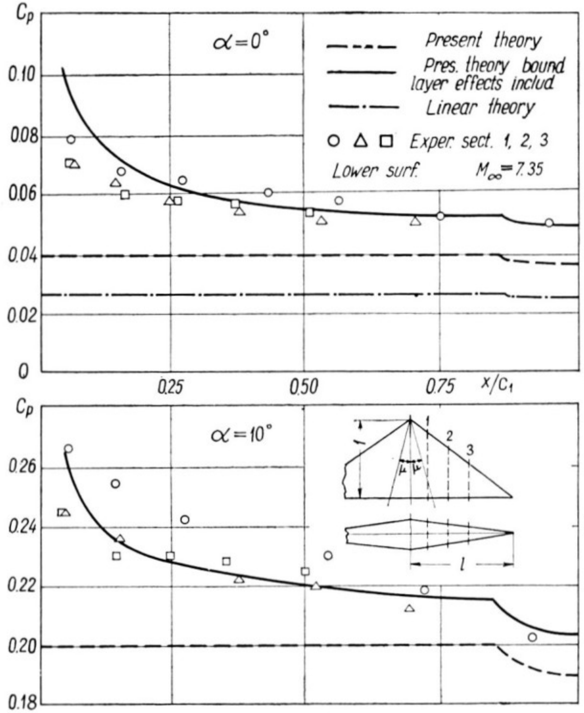

FIG. 12 — Comparison of theory and experiment. Pressure distribution on a delta wing with constant slope at $M_\infty = 7 \cdot 35$, various incidences and wing sections. Experimental values are given according to G. P. Wilby[40]

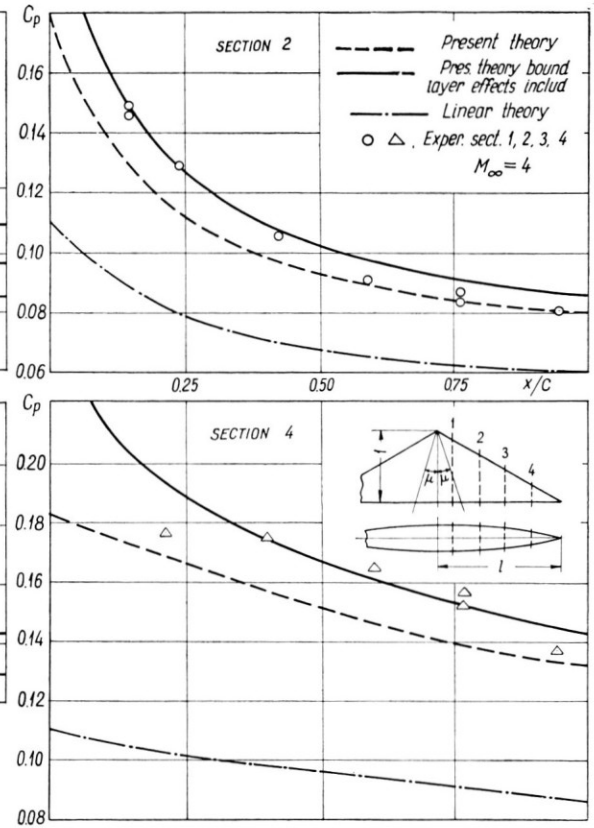

FIG. 13 — Comparison of theory and experiment. Pressure distribution on a conical delta wing with parabolic slope variation, at $M_\infty = 4$ in various sections. Experimental values are given according to M. Landahl, G. Drougge, B. Beverly[23]

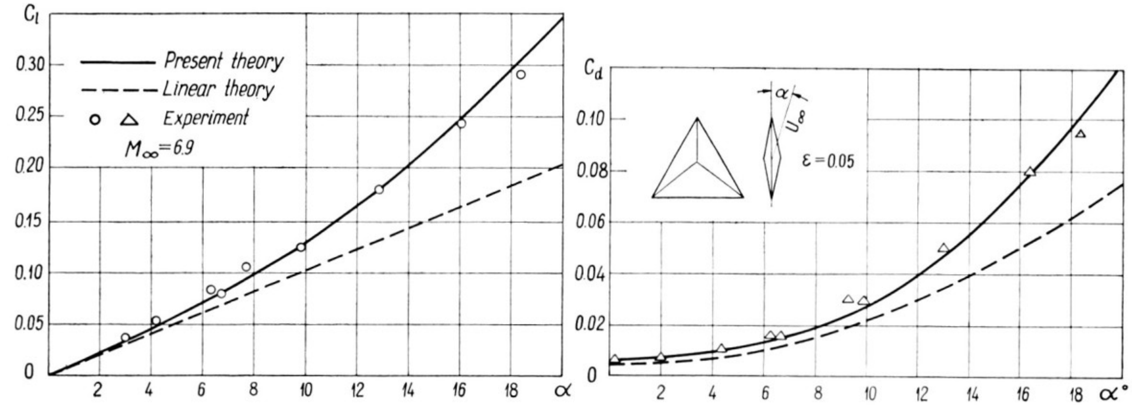

FIG. 14 — Comparison of theory and experiment. Lift (C_l) and wave drag (C_d) coefficients in terms of the incidence for a diamond-shaped delta wing at $M_\infty = 6{\cdot}9$. Experimental values are given according to C. H. McLellan[28]

been also taken of the influence of the boundary layer since this latter one alters the shape of the wing surface.

We shall remark however that, for negative angles, the agreement is less fair, owing to the thickening of the boundary layer which delays the expansion or diminishes it.

In Fig. 13 we have reproduced the experiments carried out by Landahl, Drougge, Beverly[23] at $M_\infty = 4$ on a wing of symmetrical thickness having a parabola arc as directrix. In this case too, the agreement between theory and experiment is very good. As a matter of fact, a series of other experiments confirm the theory presented above.

Obviously, the overall aerodynamic characteristics (lift, wave drag coefficients etc.) also observe this confirmation, as indicated in Fig. 14, in which the tests taken from ref. 28 have been made at $M_\infty = 6{\cdot}9$.

The theories and formula which were presented above have been confirmed experimentally, such that the results obtained can be applied with a good approximation to the calculus of pressures on wings.

7. Conclusions

The fair agreement between the values calculated according to the theoretical formulae and the experiments carried out over a wide range of incidences and Mach numbers enables one to conclude that the formulae established above are valid for all the stages of the motion around lifting systems.

References

(1) Beshkine, L., 'Forces aérodynamiques agissant sur les surfaces portantes aux vitesses supersoniques.' *Les Cahiers d'Aérodynamique*, **6**, 1947.
(2) Carafoli, E., 'High-Speed Aerodynamics.' *Editura Tehnică*, București, 1956. Also in Russian: *Aerodynamika bolshih skorostey*. Izd-vo Akad. Nauk S.S.S.R., Moscow, 1960.
(3) Carafoli, E., 'The hydrodynamic character of the solutions of conical flow applied to the theory of polygonal wings' (in Romanian). *Comunicările Acad. R.P.R.*, t. II, no. 9–10. 1952.
(4) Carafoli, E., 'On the lift and drag of the triangular wing at supersonic speeds' (in Romanian). *Comunicările Acad. R.P.R.*, t. IV, no. 5–6, 1954.
(5) Carafoli, E., Horovitz, B., 'L'écoulement supersonique autour d'une aile angulaire à disque axial. *Revue de Mécanique Appliquée*, I, 1, 1956, Bucharest.
(6) Carafoli, E., Horovitz, B., 'Homogeneous flows of higher order round a cruciform wing, with application to harmonic oscillations.' *The XIth International Congress of Applied Mechanics*, Brussels, Sept. 1956.
(7) Carafoli, E., Mateescu, D., 'L'écoulement autour du système portant aile-fuselage conique.' *Revue de Mécanique Appliquée*, IV, 3, 1959, Bucharest.

(8) Carafoli, E., 'Theory of simple or cruciform wings in supersonic flow' (in Russian). *Inzhenernyi Sbornik*, XXVII and XXVIII, 1960.

(9) Carafoli, E., Mateescu, D., 'General method for determining the wing-conical body interference in supersonic flow.' *Revue de Mécanique Appliquée*, V, **2**, 1960, Bucharest.

(10) Carafoli, E., Mateescu, D., 'High-order conical flow around a delta wing fitted with conical body.' *Proc. Seminar Aer. Sci.* (Nov. 27–Dec. 2, 1961, Bangalore, India), pp. 249–69.

(11) Carafoli, E., 'On a unitary formula for compression-expansion in supersonic-hypersonic flow.' *Revue de Mécanique Appliquée*, VII, **5**, 1962, Bucharest.

(12) Carafoli, E., Berbente, C., 'Determination of the characteristics of aerodynamic profiles in supersonic-hypersonic flow, considering the pressure losses through shock waves.' *Revue de Mécanique Appliquée*, IX, **5**, 1964, Bucharest.

(13) Carafoli, E., 'Extension of conical motions to quasi-conical motions.' *Revue de Mécanique Appliquée*, X, **2**, 1965, Bucharest.

(14) Carafoli, E., Berbente, C., 'Calculation of pressures and aerodynamic characteristics of delta wings in supersonic-moderate hypersonic flow.' *Revue de Mécanique Appliquée*, XI, **3**, 1966, Bucharest.

(15) Carafoli, E., 'Sur la théorie des ailes à bords d'attaque courbes en courant supersonique.' *L'Aerotecnica*, Vol. XLVI, no. 3, 5.9.66.

(16) Carafoli, E., Staicu, St., 'On the theory of thin delta wings with flow separation at the leading edges' (in Romanian). *Studii si Cercétări de Mecanică Aplicată*, XXII, **4**, 1966, Bucharest.

(17) Fénain, M., 'La théorie des écoulements à potentiel homogène et ses applications au calcul des ailes en régime supersonique.' *Prog. Aer. Sci.*, I, Pergamon Press, London, 1961.

(18) Garrick, I. E., 'Aerodynamic theory and its applications to flutter.' AGARD, Rep. 34, 1956.

(19) Germain, P., 'La théorie des mouvements coniques et ses applications à l'aérodynamique.' Thése, Paris, 1948, ONERA, 34 (1949).

(20) Germain, P., 'La théorie des mouvements homogènes et son application au calcul de certaines ailes delta en régime supersonique.' *Rech. Aero.* **7,** 3–16, 1949.

(21) Küchemann, D., 'A non-linear lifting-surface theory for wings of small aspect-ratio with edge separation.' R.A.E. Rep. 2540, April 1955.

(22) Lance, G. N., 'Some problems of supersonic flight.' Ph.D. Thesis, University of London. 1952.

(23) Landahl, M., Drougge, G., Beverly, B., 'Theoretical and experimental investigations of second-order supersonic wing-body interference. *J.A.S.*, **9,** 1960.

(24) Legendre, R., 'Nappes des tourbillons en cornets.' *CRAcSc* Janv. 1964.

(25) Legendre, N., 'Nappes des tourbillons déferlant des bords d'attaques des ailes en delta.' ONERA. Communic. présentée au Symposium IUTAM, Ann-Arbor, 6–10 Juillet 1964.

(26) Lighthill, M. J., 'Oscillating airfoils at high Mach number. *J.A.S.*, **6,** June 1953.

(27) Mangler, K. W., Smith, J. H., 'A theory of the flow past a slender delta wing with leading edge separation.' *Proc. Roy. Soc.*, May 1959.

(28) McLellan, C. H., 'Exploratory wind-tunnel investigation of wings and bodies at $M_\infty = 6{\cdot}9$.' *J.A.S.*, **18,** 1951.

(29) MICHAEL, W. H., 'Flow studies on flat-plate delta wings at supersonic speed. NACA TN 3472, July 1955.

(30) MILES, J. W., 'On harmonic motion at supersonic speeds.' *J.A.S.*, **16,** 378, 1949.

(31) MILES, J. W., 'On the reduction of unsteady supersonic flow problems to steady flow problems.' *J.A.S.* 17, 64, 1950.

(32) ORMEROD, O. A., HAYNES, A. B. — *see* Squire, L. C. 35, 480.

(33) ROY, M., 'Sur la théorie de l'aile delta. Tourbillons d'apex et nappes en cornets.' *CRAcSc*, R.A. no. 56. Fév. 1957.

(34) ROY, M., 'Remarques sur l'écoulement autour des ailes en flèche' *Zeits. f. ang Math. W. Phys.*, p. 554. 1958

(35) SQUIRE, L. C., 'The estimation of the non-linear lift of delta wings at supersonic speeds.' *JRAeS*. Aug. 1963.

(36) SQUIRE, L. C.' 'The characteristics of some slender cambered gothic wings at Mach numbers from 0·4 to 2·00. R & M. 3370, 1964.

(37) TEMPLE, G. *Modern Developments in Fluid Dynamics*. High Speed Flow. Edited by L. Howarth. Oxford Univ. Press, 1953.

(38) WARD, G. N. *Linearized Theory of Steady High-Speed Flow*. Cambridge University Press, 1955.

(39) WERLÉ, H., FIANT, C., 'Visualisation hydrodynamique de l'écoulement à basse vitesse autour d'une maquette d'avion du type "Concorde".' ONERA, *La Rech. Aérospatiale*, no. 102, Sept.–Oct. 1964.

(40) WILBY, P., 'An experimental and theoretical investigation at high Mach numbers.' FFA Report 91, 1962.